The Application of Negative Feedback to Frequency-Modulation Systems *

By J. G. CHAFFEE

Negative feedback can be applied to a frequency-modulation receiver of superheterodyne type by causing a portion of the output voltage to frequency-modulate the local oscillator in such phase as to reduce the output signal. As a consequence of this arrangement the effective frequency modulation of the intermediate wave is diminished by the feedback factor. This reduction is accompanied by a decrease in noise and distortion. Restoration of the original signal level by increasing the degree of modulation at the transmitter brings about a corresponding increase in signal-to-noise ratio provided the disturbance is not too great, while distortion ratios are improved to about the same extent. These effects are treated analytically for the case where the disturbance level is sufficiently low to permit simplifying assumptions to be made. The results are in general agreement with observations made on an experimental laboratory system.

Comparing the feedback system with a frequency-modulation system using amplitude limitation, the ratio of signal level to noise level in the absence of modulation is identical in two systems. During modulation the noise level increases in the feedback system by an amount depending upon the ratio of the effective frequency shift of the intermediate-frequency wave to the signal band width. By keeping this ratio small, the increase in noise during modulation can be made relatively unimportant.

In cases where the disturbance level is high, phenomena have been observed which are very similar to those encountered when amplitude limitation is used.

Introduction

THIS paper describes a method for improving the performance of receivers designed to receive frequency-modulated waves. In its broader aspects this method can be described as the application of the principle of negative feedback to a superheterodyne frequency-modulation receiver. In its details the application of the feedback principle necessitates the use of a rather unusual circuit arrangement. This circuit differs from that of the simple feedback system in that the voltages fed back are not of the same frequency as those applied

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to the input of the receiver, and are caused to influence the response of the system by modifying the performance of the modulator.

In the ordinary feedback amplifier a part of the output voltage is carried back to the input and there combined with the applied voltage. The result is to modify the output and if the gain of the system is thereby reduced the feedback is said to be negative. The many advantages which result from negative feedback have been described by Black [1] and are coming to be more generally appreciated. The present paper deals with a method for adapting this principle to a frequency-modulation receiver and will show an example of its application to an experimental system in the laboratory.

General Discussion

Method of Applying Feedback

Consider a frequency-modulation receiver in which the incoming wave is combined with the output of a local oscillator in a modulator to produce a wave of intermediate frequency. This is then amplified, converted into an amplitude-modulated wave, and finally detected. The frequency of the intermediate wave is equal to the instantaneous difference in the frequencies of the incoming carrier and the local oscillator. So long as the frequency of this oscillator remains fixed the intermediate wave will be frequency-modulated in exact correspondence with the incoming wave. Suppose now that the local oscillator is frequency-modulated from a source of the same frequency and phase as that applied to the transmitter. As the index of modulation at the local oscillator is increased from zero the extent to which the intermediate wave is modulated will diminish since its instantaneous frequency is equal to the difference in the frequencies of the two sources. It then follows that if these two devices are modulated to the same extent the difference frequency will become constant and the output of the system will be zero. Finally a further increase in modulation of the local oscillator will cause the intermediate wave to be modulated with a 180-degree phase reversal.

This process can be readily analyzed as follows: Assume the oscillator at the transmitter to have been frequency-modulated by the signal wave

$$e = E_1 \cos pt. \tag{1}$$

The voltage delivered to the modulator by the incoming wave will be

$$A \cos (\omega_1 t + x_1 \sin pt + \phi_1) \tag{2}$$

[1] H. S. Black, "Stabilized Feedback Amplifiers," *Elec. Engg.*, vol. 53, pp. 114–120, January 1934.

where $x_1 = \Delta\omega_1 \div p = \rho E_1 \div p$, and $\Delta\omega_1$ is 2π times the maximum frequency shift. The local oscillation impresses the wave

$$B \cos (\omega_2 t + x_2 \sin pt + \phi_2) \tag{3}$$

where

$$x_2 = \Delta\omega_2 \div p.$$

Application of these waves to a square-law modulator will yield a difference frequency wave proportional to

$$AB \cos [(\omega_1 - \omega_2)t + (x_1 - x_2) \sin pt + \phi_1 - \phi_2] \tag{4}$$

for the case where $\omega_1 > \omega_2$, or when the reverse is true

$$AB \cos [(\omega_2 - \omega_1)t - (x_1 - x_2) \sin pt + \phi_2 - \varphi_1]. \tag{5}$$

In either case the resultant modulation index of the intermediate wave is the numerical difference of the original indexes, the difference in sign

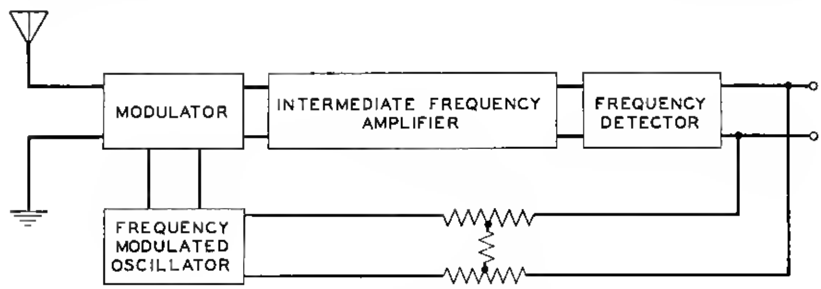

Fig. 1—Basic feedback circuit.

in the two cases signifying that the detected outputs will be of opposite phase. If $x_1 = x_2$ the modulation is reduced to zero, and if $x_2 > x_1$ modulation reappears with a phase reversal. It might be noted that if x_2 were originally made negative, thus causing the two oscillators to be frequency-modulated in opposite phase, the apparent modulation of the incoming wave could be increased indefinitely.

Suppose, now, that instead of frequency-modulating the local oscillator from an independent source, the equivalent is accomplished in a practical way. For this purpose a voltage from the output of the receiver is impressed upon the local oscillator as shown in Fig. 1. The transmitted wave will then have a modulation index $x_1 = \rho_1 E_1 \div p$, while the local oscillator, being acted upon by a portion of the output voltage E_0, will have an index $x_2 = k\rho_2 E_0 \div p$. If the frequency detector [2] is assumed to be linear the amplitude of the detected output

[2] The term *frequency detector* is used in this paper to designate the combination of conversion circuit and amplitude detector. A more extended discussion of modulation and detection is given in Appendix A.

will be proportional to the product of the amplitudes A and B of the incoming and local oscillator waves, the resultant index of the intermediate wave, and the slope factor a_1. Thus we can write the output voltage amplitude

$$E_0 = \alpha a_1 AB(x_1 - x_2)p = \alpha a_1 AB(\rho_1 E_1 - k\rho_2 E_0). \tag{6}$$

Therefore

$$E_0 = \frac{\alpha a_1 AB\rho_1 E_1}{1 + \alpha k a_1 AB\rho_2} . \tag{7}$$

Setting $\alpha a_1 AB = \mu$ and $k\rho_2 = -\beta$ we obtain the familiar form encountered in the analysis of feedback amplifiers

$$E_0 = \frac{\mu(\rho_1 E_1)}{1 - \mu\beta} . \tag{8}$$

Without feedback the output of the system is merely $\mu(\rho_1 E_1)$. The feedback factor $1 + \alpha a_1 kAB\rho_2$ is a measure of the extent to which the over-all gain of the system has been modified by feedback. If this factor is greater than unity the feedback is negative, while if k is made negative by reversing the feedback connections the effect is regenerative, and instability is encountered when the factor becomes zero.

It will be noted that when $\alpha a_1 kAB\rho_2 \gg 1$, (7) becomes

$$E_0 = \frac{\rho_1 E_1}{k\rho_2} . \tag{9}$$

Thus for large amounts of feedback, the output signal becomes independent of such factors as fading of the incoming wave, variations in the local oscillator voltage, or changes in detector efficiency. Hence automatic gain control is secured. This feature is equivalent to that found in ordinary feedback amplifiers in that for large amounts of feedback the over-all gain becomes independent of variations in the performance of the amplifier proper.

Reduction of Noise

The application of negative feedback in the manner described brings about a reduction in signal level by decreasing the effective modulation of the received wave. It then becomes possible to increase the modulation level at the transmitter to a corresponding degree and thus to restore the output signal to its former value. This process is made possible through the use of frequency rather than amplitude modulation since the permissible degree of modulation is then deter-

mined by the receiver characteristics. It will be shown that feedback also reduces the noise level at the output of the receiver, provided that the disturbance is not too great. Thus when the modulation level is raised to offset the effect of feedback an improvement in signal-to-noise ratio is realized.

The mechanism by which noise is reduced can be described qualitatively as follows: Noise at the output terminals of the receiver is caused to frequency-modulate the intermediate wave in such fashion as to produce, upon detection, a component which tends to cancel that which would exist in the absence of feedback. An analysis of this process for the case where the carrier is large compared with the disturbance responsible for the noise is developed [3] in Appendix B. It is assumed that the disturbance can be represented by a continuous spectrum of sinusoidal voltages of equal amplitude but phased at random. Impressed along with the disturbance is the signaling wave (2). Then if N^2 is the mean disturbing power per unit of band width in the vicinity of the carrier frequency, and r_1 is the resistance of the input circuit, it is shown that the output noise power is [4]

$$P_N = \frac{2N^2 r_1}{F^2}\left[a_0^2 + \frac{a_1^2 \Delta\omega^2}{2F^2} + \frac{a_1^2 q_a^2}{3}\right] q_a \tag{10}$$

where a_0 and a_1 are, respectively, the gain and slope factor of the intermediate amplifier and conversion system as defined by (47), and q_a represents the upper limit of frequency response of the output circuit, or the upper limit of audibility as the case may be. F is the feedback factor $(1 - \mu\beta)$. The corresponding signal power is

$$P_s = \frac{A^2 a_1^2 \Delta\omega^2}{2F^2}. \tag{11}$$

The reduction in signal level occasioned by feedback can be offset by increasing the frequency shift of the transmitted wave. If it is increased so as to have the value $\Delta\Omega = F\Delta\omega$ then the shift of the inter-

[3] An analysis of the effect of feedback upon noise in this system was first developed by J. R. Carson by methods similar to those used in "Variable Frequency Electric Circuit Theory with Application to the Theory of Frequency Modulation," Carson and Fry, *Bell Sys. Tech. Jour.*, vol. 16, pp. 513–540, October 1937. This has been embodied in a paper by Mr. Carson entitled, "Frequency Modulation: Theory of the Feedback Receiving Circuit," published in this issue of the *Bell Sys. Tech. Jour.* Carson's treatment is more general in that an arbitrary signal wave is postulated whereas the analysis given in Appendix B is restricted to a sinusoidal signal wave. The methods used here are more elementary and may therefore appeal to a somewhat wider audience.

[4] The expressions for signal and noise power used in this section are relative. Factors determining their absolute magnitude are given in the Appendix. In all cases the symbol $\Delta\omega^2$ is to be taken as signifying $(\Delta\omega)^2$.

mediate-frequency wave will be restored to its original value $\Delta\omega$ and the signal level will remain unchanged. Then the noise power becomes

$$P_N = \frac{2N^2r_1}{F^2}\left[a_0^2 + \frac{a_1^2\Delta\Omega^2}{2F^2} + \frac{a_1^2q_a^2}{3}\right]q_a \qquad (12)$$

which can be written

$$P_N = \frac{2N^2r_1}{F^2}\left[a_0^2 + \frac{a_1^2\Delta\omega^2}{2} + \frac{a_1^2q_a^2}{3}\right]q_a. \qquad (12a)$$

The noise-to-signal power ratio is improved by the factor F^2, since

$$\frac{P_N}{P_s} = \frac{1}{F^2}\frac{4N^2r_1}{A^2}\left[\frac{a_0^2}{a_1^2\Delta\omega^2} + \frac{1}{2} + \frac{q_a^2}{3\Delta\omega^2}\right]q_a. \qquad (13)$$

Of the factors in (12) the first is the result of modifications of the amplitude of the incoming wave by the disturbance. Although subject to reduction by feedback it can be balanced out completely by the use of differentially connected frequency detectors having slope factors a_1 and $-a_1$. The second term is dependent upon the degree of modulation of the intermediate wave. It is usually of less consequence in its effect upon the listener. The remaining term is the result of phase modulation of the signal wave by the disturbance. Under the condition that the output signal is held constant by increasing the transmitted band, all terms which contribute to the noise level in a given case are reduced alike by feedback.

If differential frequency-detection is employed (12a) becomes

$$P_N = \frac{2N^2r_1a_1^2}{F^2}\left[\frac{\Delta\omega^2}{2} + \frac{q_a^2}{3}\right]q_a. \qquad (14)$$

During non-signaling periods the first term becomes zero. Hence during periods of modulation the background noise power is increased by the factor

$$1 + \frac{3}{2}\frac{\Delta\Omega^2}{F^2q_a^2} = 1 + \frac{3}{2}\frac{\Delta\omega^2}{q_a^2}\,. \qquad (15)$$

If conditions are such that the maximum shift experienced by the intermediate-frequency wave is numerically equal to q_a, then the noise level will be increased by 4 decibels during periods of full modulation. In the experimental system to be described the ratio of $\Delta\omega$ to q_a was allowed to attain a value of 1.75, resulting in a maximum increase of 7.5 decibels.

In order to secure large noise reduction it is necessary to produce a frequency shift in the transmitted wave much greater than the signal band width. Thus, in common with frequency-modulation systems employing amplitude limiters,[5] this advantage is secured at the expense of band width. In this connection it is of interest to compare amplitude limitation and feedback systems on the basis of equal width of transmitted band. Hence it will be assumed that in each case the transmitted wave is modulated to the extent of $\pm\, \Delta\Omega = \pm\, F\Delta\omega$. In Fig. 2 are shown idealized characteristics of conversion systems which might be used in the two systems. The adjustment shown in Fig. 2(a) is suitable for use with the limiter system. With the feedback system that shown in Fig. 2(b) would be necessary to secure the same percentage of amplitude modulation after conversion. This represents

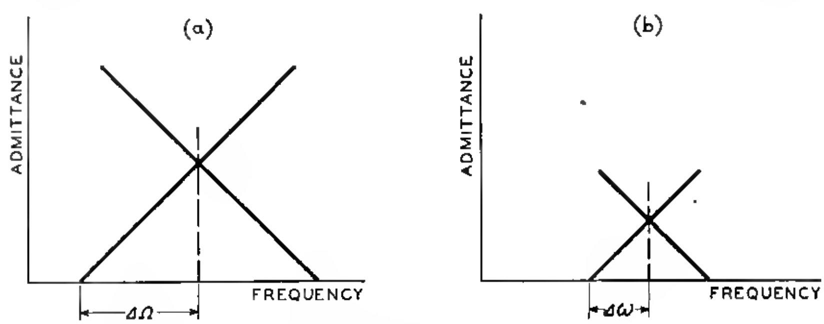

Fig. 2—Idealized conversion-system characteristics for (a) limiter system, (b) feedback system.

the minimum band width which could be provided in the conversion system with feedback, though several considerations make it desirable to use an adjustment lying somewhere between the two shown. The manner of tuning or the slope factors assumed in each case are immaterial to the present comparison provided that, in either system, the linear portion of the characteristic is of sufficient extent to effect proper conversion of the intermediate-frequency wave.

The noise-to-signal power ratio obtainable with the feedback system will be that given by (13) with the first term omitted since it is balanced out by the push-pull arrangement. Thus

$$\frac{P_N}{P_s} = \frac{4N^2r_1}{F^2A^2}\left[\frac{1}{2} + \frac{q_a^2}{3\Delta\omega^2}\right] q_a. \tag{16}$$

[5] E. H. Armstrong, "A Method of Reducing Disturbances in Radio Signaling by a System of Frequency Modulation," *Proc. I. R. E.*, vol. 24, pp. 689–740, May 1936.

In the system corresponding to Fig. 2(a), the signal power will be

$$\frac{A^2 a_1^2 \Delta\Omega^2}{2}. \tag{17}$$

Equation (10) can be used to determine the noise power level for a frequency-modulation system without amplitude limitation by setting $F = 1$. If balanced detection is used the term in a_0 becomes zero. It has been shown by Carson and Fry [3] that the addition of an ideal limiter removes all terms but the third, with either single or balanced detectors. Hence for the limiter system the noise ratio becomes

$$\frac{P_N}{P_s} = \frac{4N^2 r_1}{A^2}\left(\frac{q_a^3}{3\Delta\Omega^2}\right). \tag{18}$$

Since $\Delta\Omega = F\Delta\omega$ this can be put in a form similar to (16)

$$\frac{P_N}{P_s} = \frac{4N^2 r_1}{F^2 A^2}\left(\frac{q_a^2}{3\Delta\omega^2}\right) q_a. \tag{19}$$

Comparing (16) and (19) it is seen that the noise ratio in the feedback system is greater than that for the limiter system by the factor (15). This is a consequence of the increase in noise level which occurs during modulation in the former system. The ratio of noise level during non-signaling periods to signal level is identical in the two systems.

While the noise increment which appears during modulation is usually not of great consequence from a practical standpoint, it can be reduced by increasing the feedback factor beyond the point dictated by the signal band which it is permissible to transmit. In previous discussions it has been assumed that the application of a given amount of negative feedback is to be accompanied by a corresponding increase in modulation level at the transmitter. In this way the modulation of the intermediate frequency wave is kept constant so as to maintain a fixed signal level as the band width of the transmitted wave is increased. Having arrived at a limiting value of band spread the feedback factor can be increased further. Suppose that modulation of the transmitter is to be limited to a value of $\Delta\Omega = F_1\Delta\omega$, but that the feedback applied to the receiver is made to exceed F_1 by a factor which we shall call F_2. Then the actual feedback factor will be F_1F_2 and we have

$$P_s = \frac{A^2 a_1^2 \Delta\omega^2}{2F_2^2} \tag{20}$$

$$P_N = \frac{2N^2 r_1}{F_1^2 F_2^2}\left[\frac{a_1^2\Delta\omega^2}{2F_2^2} + \frac{a_1^2 q_a^2}{3}\right] q_a \tag{21}$$

giving

$$\frac{P_N}{P_s} = \frac{4N^2 r_1}{A^2 F_1^2}\left[\frac{1}{2F_2^2} + \frac{q_a^2}{3\Delta\omega^2}\right] q_a. \tag{22}$$

Thus the additional feedback represented by the factor F_2 is directly effective against the noise increment accompanying modulation. Reduction of this increment brings about a still closer correspondence between the limiter and feedback systems as is seen by setting $F = F_1$ in (19) and comparing with (22).

The above discussion and the analysis given in Appendix B are based upon the assumption that the carrier amplitude is large compared with that of the disturbance. A rigorous analysis, applicable to the case where this ratio is unrestricted, becomes exceedingly involved. However, a rough indication of what is to be expected in the presence of a high level of disturbance can be obtained quite simply from (52) developed in Appendix B. Assuming that modulation is not present this can be put in the simple form

$$\sigma = \frac{1}{F}\frac{Q'(a_0 + a_1\omega_n)}{1 + \frac{Q'}{A'}\left(\frac{F-1}{F}\right)\cos\omega_n t.}\cos\omega_n t. \tag{52a}$$

When $Q' \ll A'$ the wave form of the output noise produced by a single element of disturbance is very closely a sinusoid. However, when Q' and A' become comparable in magnitude the output wave becomes badly peaked when $\omega_n t = n\pi$. While the above expression is only a very rough approximation under these conditions, a plot of the wave form so obtained exhibits all of the essential characteristics of the curves given by Crosby in a recent paper [6] dealing with noise in frequency-modulation systems using amplitude limitation. These curves show a similar peaking of the output-noise wave form when the ratio of carrier to disturbance amplitude is in the vicinity of unity. The description given by Crosby of the manifestations of this phenomenon observed in an experimental system applies rather closely to what has been found in the feedback system. A more detailed account will be found in a later section.

Examination of (52a) shows that the output wave can assume very large and even infinite peak values when Q' and A' are approximately equal. The existence of high peak values of noise implies both a large instantaneous deviation in the frequency of the intermediate wave, and a conversion-circuit characteristic of unlimited extent. The finite

[6] Murray G. Crosby, "Frequency Modulation Noise Characteristics," *Proc. I. R. E.*, vol. 25, pp. 472–514, April 1937. The curves referred to are given in Fig. 4 of the above paper.

limits of the characteristic of the over-all intermediate-frequency system have the effect of holding the maximum peaks of noise to a value equal to the highest signal peaks obtainable in the absence of the disturbance. Furthermore, the existence of high noise peaks in the presence of modulation can result in the momentary assumption by the instantaneous intermediate frequency of values outside of the region to which the system is normally responsive. Thus the output signal will appear to be chopped by the higher noise peaks, and as a consequence its energy content will be considerably reduced.

The above effects are, of course, present in systems using limiters and have already been discussed in greater detail by Crosby.[6]

Distortion Reduction

One of the chief benefits which can be realized through the use of negative feedback is the reduction of non-linear distortion products generated in the forward branch of the system. While the distortion in properly designed amplifiers is sufficiently low for many purposes, cases frequently arise in which the requirements are much more severe. In an amplifier which is to handle several channels in a high grade multiplex system, the distortion products should be of the order of 60 decibels below the fundamental of the output. This degree of excellence is most readily obtained by using negative feedback.

In radio systems designed for multiplex service it is of equal importance that the distortion level be kept at a correspondingly low level if crosstalk is to be avoided. It is therefore of interest to inquire into the manner in which distortion is modified in the present feedback system.

An analysis of the effect of feedback upon distortion is given in Appendix A. If the transmitter is modulated with a signal wave $S = S(t)$ so that its instantaneous frequency becomes

$$\omega + \rho_1 S \tag{23}$$

then, in the presence of non-linearity in the receiver, the output of the system can be written as a power series in the variable frequency term $\rho_1 S$. Thus for the first three orders we shall have

$$\sigma = \alpha AB[b_1\rho_1 S + b_2(\rho_1 S)^2 + b_3(\rho_1 S)^3]. \tag{24}$$

If feedback is applied without altering the modulation level at the transmitter it is shown that the above series becomes

$$\sigma_F = \alpha AB\left(\frac{b_1}{F}\rho_1 S + \frac{b_2}{F^3}(\rho_1 S)^2 + \frac{1}{F^4}\left[b_3 - \frac{2b_2^2}{b_1}\left(\frac{F-1}{F}\right)\right](\rho_1 S)^3\right). \tag{25}$$

When the feedback factor F is large this can be written

$$\sigma_F = \alpha AB\left(\frac{b_1}{F}\rho_1 S + \frac{b_2}{F^3}(\rho_1 S)^2 + \frac{1}{F^4}\left[b_3 - \frac{2b_2^2}{b_1}\right](\rho_1 S)^3\right). \quad (26)$$

Upon increasing the modulation by the factor F so as to restore the original level of the fundamental, the output becomes

$$\sigma_F' = \alpha AB\left(b_1\rho_1 S + \frac{1}{F}\left[b_2(\rho_1 S)^2 + \left(b_3 - \frac{2b_2^2}{b_1}\right)(\rho_1 S)^3\right]\right). \quad (27)$$

Second order distortion products are reduced with respect to the fundamental level by the feedback factor. Third (and higher) order products are modified to an extent depending upon the relative values of the distortion coefficients and the amount of feedback. If, as can readily be the case when a balanced detecting system is used,

$$b_3 \gg \frac{2b_2^2}{b_1} \quad (28)$$

third order products are reduced in the same manner as those of second order. In any case, by applying sufficient feedback a point will be reached where a given increment in feedback will produce a corresponding reduction in all distortion products.

Equation (25) shows that the greatest improvement in distortion is obtained if the modulation level is not increased when feedback is applied. The large reductions result partly from feedback and in part from the fact that the system is operating at reduced percentage of modulation. Under this condition there is no improvement in background noise ratio, though the noise increment which takes place during modulation is diminished; see (10) and (11). Depression of both noise and distortion, but with greater emphasis upon the reduction of the latter, can be effected by raising the modulation level by an amount somewhat less than the feedback factor. This procedure has already been discussed in connection with (22) which gives the resulting noise-to-signal power ratio. Under similar conditions we have, from (25),

$$\sigma_F'' = \alpha AB\left(\frac{b_1}{F_2}\rho_1 S + \frac{1}{F_1}\left[\frac{b_2}{F_2^3}(\rho_1 S)^2 + \frac{1}{F_2^4}\left(b_3 - \frac{2b_2^2}{b_1}\right)(\rho_1 S)^3\right]\right) \quad (29)$$

when the feedback factor is large.

Equations (22) and (29) are most readily interpreted by means of Fig. 3, which illustrates the manner in which the receiver output is modified as the feedback is increased.

It is assumed that a constant signal level is maintained by an increase in modulation level up to a point corresponding to the factor F_1. Beyond this point the modulation remains fixed while the feedback

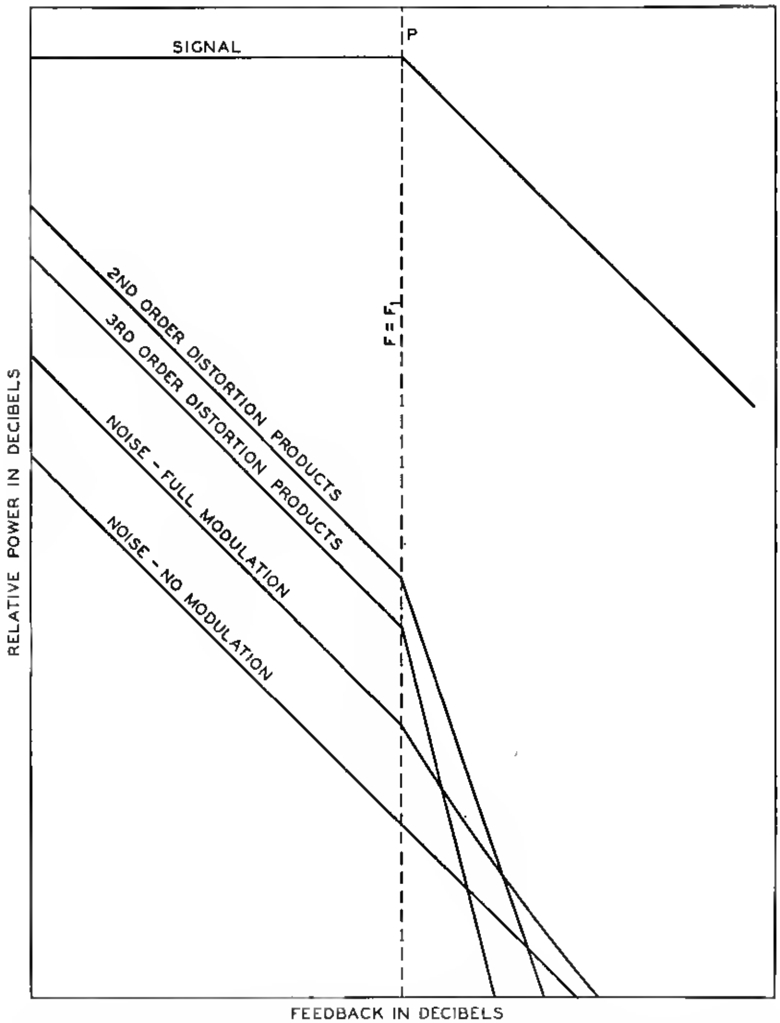

Fig. 3—Theoretical manner in which components of receiver output are modified by feedback. Modulation level at transmitter is assumed to be increased by the feedback factor up to point P, and subsequently to remain fixed.

factor is increased to F_1F_2. Since signal and noise levels have been expressed in terms of power, distortion levels are similarly expressed.

These levels, to an arbitrary decibel scale, have been plotted against decibels of feedback, given by the expression

$$10 \log F^2 = 20 \log F.$$

Balanced detection and the fulfillment of condition (28) have been assumed.

Over the region in which a constant signal output is maintained by increasing the modulation level, noise and distortion levels decrease in accordance with the feedback. The noise level during modulation continues to exceed the background noise by 4 decibels, assuming an initial frequency shift equal to the highest signal frequency to which the system is responsive.

Beyond the point at which the feedback factor has reached the value F_1, the modulation level at the transmitter is held constant. A further increase in feedback brings about a corresponding decrease in the effective percentage of modulation for the system, causing the signal level to fall in similar fashion. Distortion products now fall off still more rapidly with respect to the signal, so that an increase in feedback amounting to 1 decibel improves the second and third order distortion ratios by 2 and 3 decibels, respectively.

The ratio of signal to background or non-signaling noise remains fixed in this region in spite of the reduction in effective modulation. This ratio is that which would be obtained in a limiter system in which the same high-frequency band is transmitted. The noise increment, however, is diminished by the additional feedback and is made to approach zero.

By suitable choice of the variables F_1 and F_2 it is possible to proportion the benefits of feedback in the most advantageous manner. Thus if noise is of more consequence than distortion, modulation would be increased to the full extent of the feedback; if distortion is of primary concern, as it might well be in a multiplex system, operation as indicated in Fig. 3 would be preferable.

Experimental Results

Description of Equipment

Experimental confirmation of the principles which have been outlined has been obtained with the aid of a laboratory system shown schematically in Fig. 4. This arrangement provided a transmitter, receiver, and source of disturbance all under local control. The transmitter operated at a carried frequency of 20 megacycles. This was frequency-modulated by means of a circuit basically similar to that

described by Travis.[7] Tube noise voltage appearing at the output of a high gain radio frequency amplifier supplied the high-frequency disturbance.

At the receiver an oscillator similar to that at the transmitter served to beat down the incoming wave to an intermediate frequency of 438 kilocycles. This was applied to a three-stage amplifier having substantially uniform gain over a band of 50 kilocycles, and thence delivered to a balanced frequency detector. In addition to signal voltage, automatic-frequency-control potentials were derived from the detectors. Both were carried back to the local oscillator, but in order to permit independent control of the amount of feedback their respective paths were kept separate. In this way full frequency control could be had even with signal feedback reduced to zero.

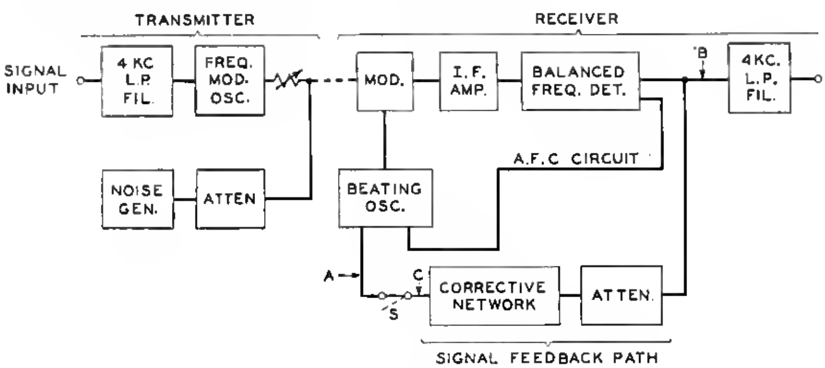

Fig. 4—Schematic of experimental feedback system.

Details of the frequency detector and feedback connections are shown in Fig. 5. The conversion system derives its characteristics from anti-resonant circuits L_1C_1 and L_2C_2, double-winding coils being used to isolate the rectifier anodes from the plate battery. One circuit is tuned to a frequency 15.4 kilocycles above the intermediate carrier frequency and the other to a corresponding point below, their characteristics intersecting at a point where the gain is approximately one half of the peak value. Detection takes place in linear rectifiers D_1 and D_2. By means of the arrangement shown, signal potentials are impressed upon the grids of amplifiers A_1 and A_2, while frequency-control voltage appears across condensers C_3, C_4. This voltage becomes zero when the receiver is correctly tuned and appears with proper polarity to

[7] Charles Travis, "Automatic Frequency Control," *Proc. I. R. E.*, vol. 23, pp. 1125–1141, October 1935.

produce correction of the frequency of the local oscillator in case of slow drifts in the frequency of either oscillator.

The use of a conversion system having peaks separated by an amount considerably exceeding the greatest frequency deviation is the result of a compromise between the readily adjustable and high impedance anti-resonant type of load circuit and others which, though more linear in their characteristics, lead to much lower gain in the conversion stage. While a peak separation of 14 kilocycles would have sufficed in view of the limitations of the transmitter, a considerably greater peak separation without corresponding increase in modulation was used. As a result that portion of the circuit characteristic actually embraced by the modulated intermediate-frequency wave

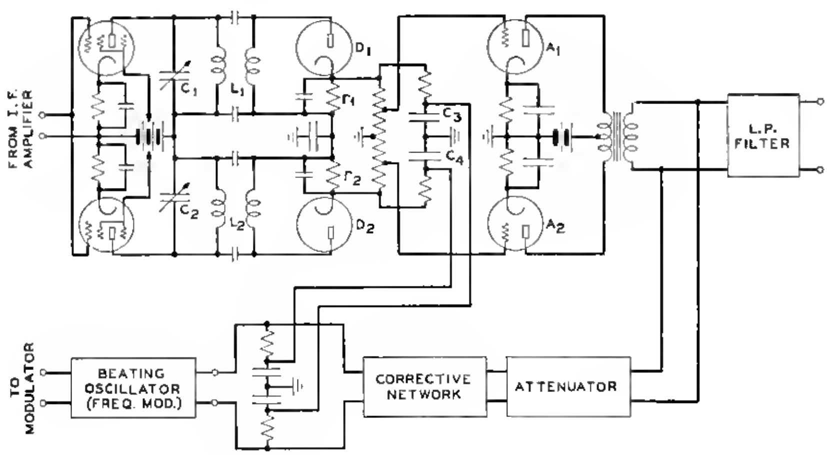

Fig. 5—Details of balanced frequency detector and feedback connections.

presented a much better approximation of a straight line than would have been possible with minimum peak separation. The penalty for adjusting the circuits in this manner is merely a loss in detecting efficiency and not an impairment of the signal-to-noise ratio. This can readily be overcome by additional audio-frequency amplification.

The signal-frequency feedback path includes an attenuator for adjusting the feedback and a corrective network for preventing singing around the feedback loop. Frequency control and feedback paths are finally combined at the modulation terminals of the local oscillator.

The necessity for the inclusion of a corrective network to modify the transmission characteristics of the feedback path is evident from Fig. 6. This shows the measured gain and phase characteristics of the receiver

alone, viewed as a voice-frequency network between points A and B in Fig. 4. This was obtained by applying signal frequencies to the modulation terminals of the beating oscillator and making observations at point B with switch S open, proper termination being provided at both sides of the break. The unmodulated transmitter served, in effect, as the beating oscillator during this measurement. At the lower signal frequencies the phase is practically 180 degrees as indicated by (4) with $x_1 = 0$. As the signal frequency is increased the phase

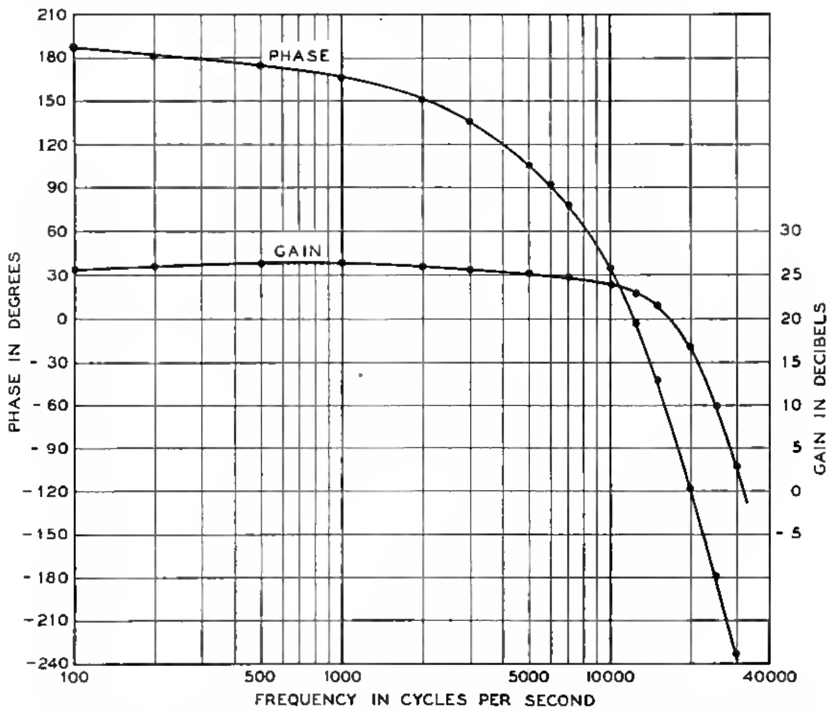

Fig. 6—Phase and gain characteristics of receiver measured between points A and B of Fig. 4 with switch S open. Transmitter in operation but not modulated.

is progressively shifted from this value. Except for that produced by the output transformer, the shift takes place within the intermediate-frequency amplifier and conversion circuits. Its magnitude is a measure of the slope of the phase-frequency characteristic of the intermediate frequency system.

The existence of positive gain at a point of zero phase shows that singing would occur if feedback connections were made directly to the beating oscillator. It was therefore necessary to reduce the gain below

unity at the point of zero phase. This was accomplished by including in the feedback path a network designed by R. L. Dietzold. The gain-frequency characteristic of this network is shown in Fig. 7. The modified loop characteristics as measured between points A and C with switch S open, and with the attenuator set for an 8-decibel loss, are given in Fig. 8. Full feedback is applied only over a band extending to 4 kilocycles, so that the range of frequencies applied to the transmitter and delivered to the listener must be restricted to this figure.

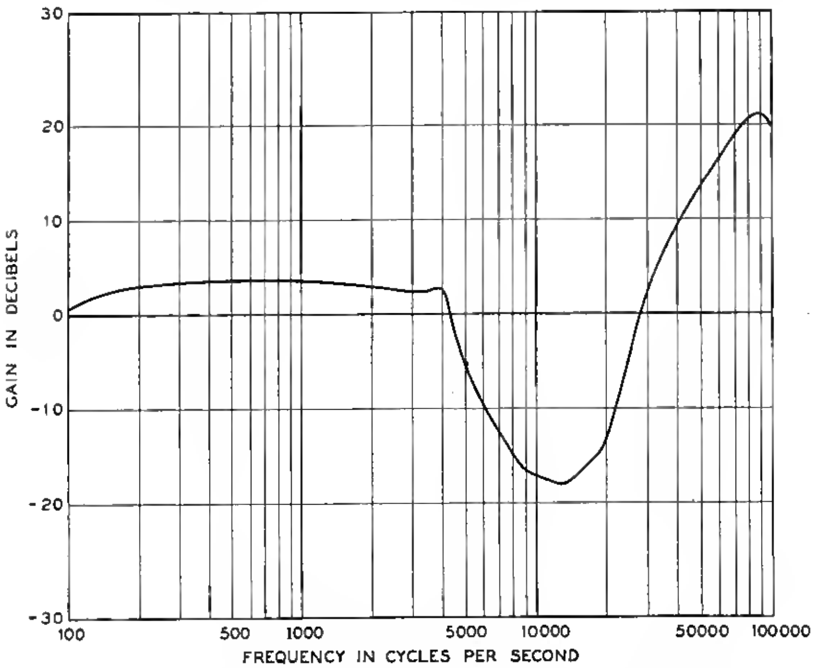

Fig. 7—Gain-frequency characteristic of corrective network inserted in signal feedback path.

The limit of stable feedback which can be realized is indicated by the difference between the loop gain within the useful band and that at the frequency corresponding to zero phase.

Distortion Measurements

The manner in which distortion levels at the output of the receiver were observed to vary with feedback is depicted in Figs. 9 to 12. In each case the modulation level for zero feedback was such as to shift the frequency of the transmitter ± 7 kilocycles at the rate of 1000

cycles per second. Figure 9 shows the effect of increasing the modulation in proportion to the feedback so as to maintain a constant output level for the fundamental. Both second and third harmonics tend to be reduced in proportion to the feedback, the improvement in third-harmonic level being 23.5 decibels for 25-decibel feedback. Failure to realize full reduction of the second harmonic is the result of distortion beginning to manifest itself in one or the other of the modulated oscillators. At the point of 25-decibel feedback the transmitter and

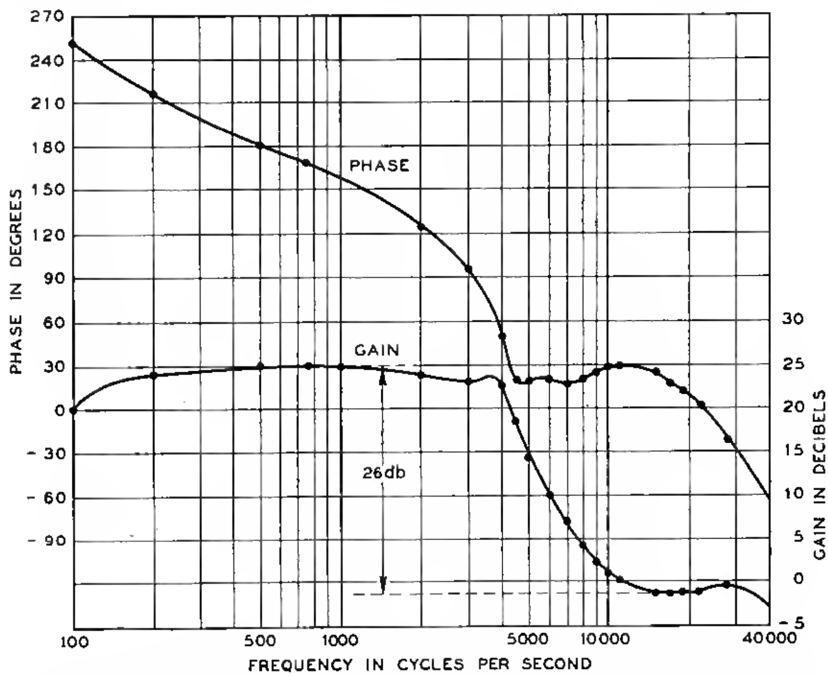

Fig. 8—Gain and phase characteristics of complete feedback loop including corrective network. Measured between points *A* and *C* of Fig. 4 with switch *S* open.

beating oscillator were being modulated to the extent of $\pm$ 124.5 and $\pm$ 117.5 kilocycles, respectively.

The curves of Fig. 10 were obtained by maintaining a constant fundamental level up to the point of 15-decibel feedback and then allowing the modulation level at the transmitter to remain constant thereafter. The results correspond rather closely with the theoretical curves of Fig. 3 and show the very rapid decrease in distortion which takes place when the modulation level remains unaltered; see (25). A more extreme example of this method of operation is shown in Fig. 11 where

modulation was left at its initial value. Harmonic levels soon reached a point beyond which they could not be measured accurately.

In a practical system the loss in signal level resulting from operation in this manner could easily be overcome by the addition of a low-distortion audio-frequency amplifier at the output of the receiver. This amplifier might well embody negative feedback of the more usual type.

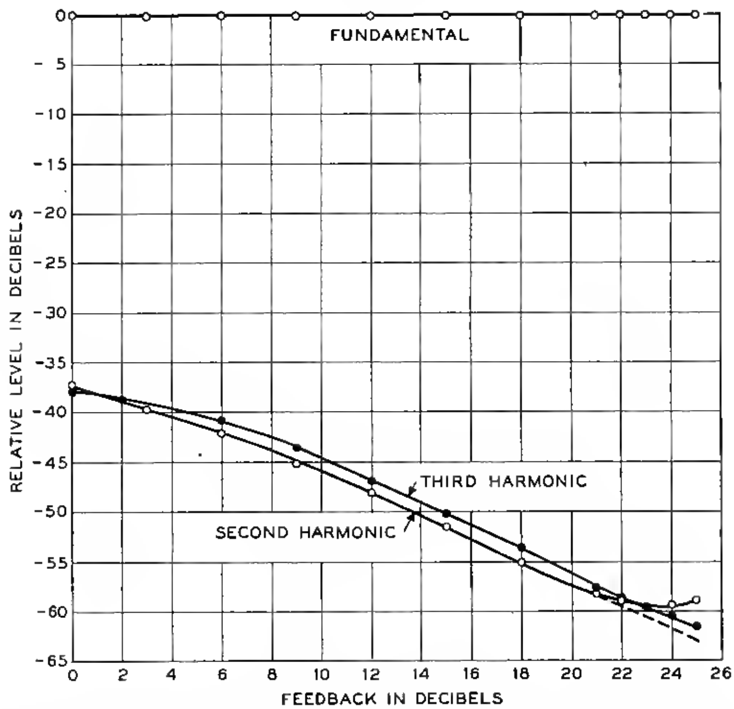

Fig. 9—Effect of feedback upon receiver distortion. Fundamental level kept constant by increasing transmitter modulation in proportion to the feedback. Modulation with no feedback = ± 7 kilocycles at 1000 cycles per second.

A composite of these distortion measurements is given in Fig. 12. Harmonic levels are plotted in decibels below the fundamental and are indicative of the improvements brought about by feedback. If it is assumed that any loss in signal is compensated by additional audio-frequency amplification, the fundamental level would be represented in all cases by the axis of abscissae.

Noise Measurements

In Fig. 13 are given the results of a series of observations of receiver output noise versus amount of feedback for a number of high-frequency

disturbance levels. Measurements were made in the absence of modulation and hence are indicative of the manner in which background noise is modified by feedback. The signal level indicated is that which could be maintained at low noise levels by increasing the modulation in proportion to the feedback, and is not significant for observations falling within or close to the shaded area, as will be explained subsequently.

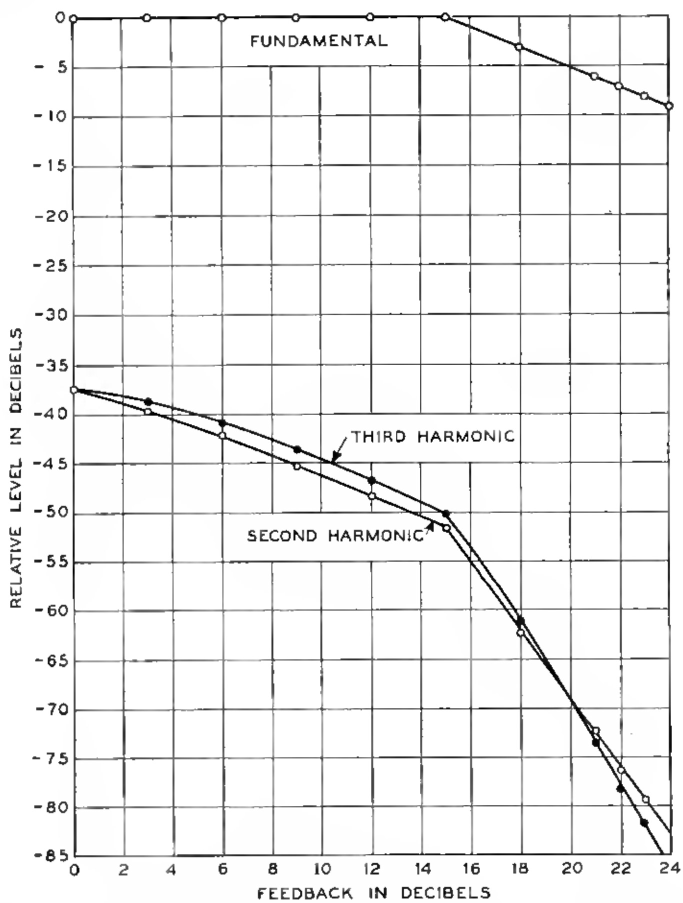

Fig. 10—Effect of feedback upon receiver distortion. Conditions same as indicated for Fig. 9 up to 15-decibel feedback; modulation held constant thereafter.

The lowest noise level shown is that generated within the receiver while the higher levels were produced by disturbances introduced from the noise generator. The relative magnitude of the effective carrier and disturbing voltages at the grids of the amplitude detector is indi-

cated on each curve. This was obtained in the following manner: With the transmitter turned off the noise attenuator was adjusted until the introduced disturbance produced the same value of rectified current as that observed when the carrier alone was applied. This determined the input level from the noise generator which produced equal root-

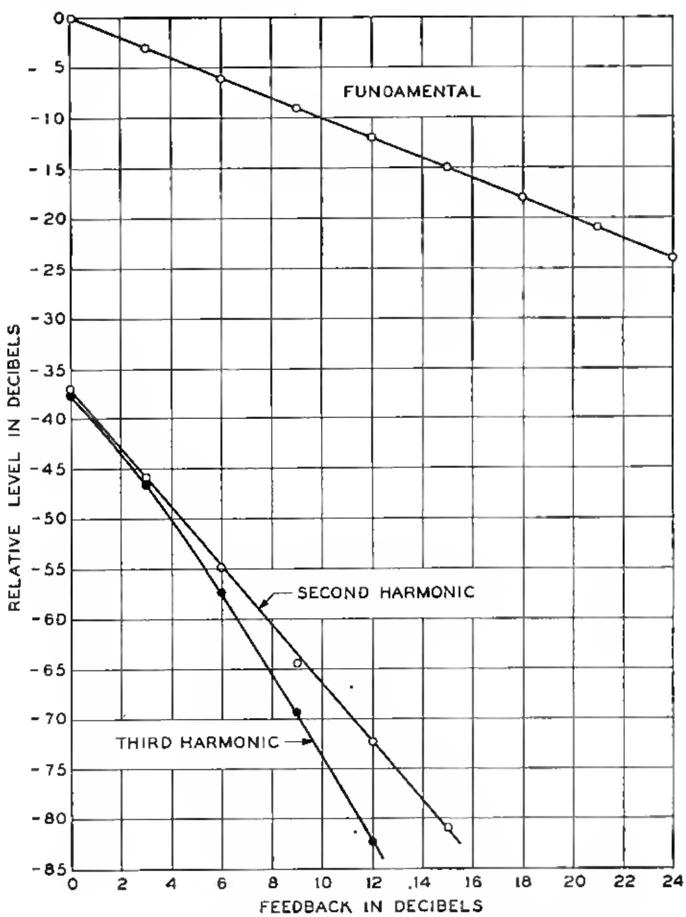

Fig. 11—Effect of feedback upon receiver distortion. Modulation held to a constant value of $\pm$ 7 kilocycles.

mean-square values of intermediate-frequency carrier and disturbance. Since at very low inputs from the noise generator the net intermediate-frequency disturbance was determined partly by tube noise generated within the receiver, a curve of output noise without feedback versus

input from the noise generator was obtained. In the region where the effect of receiver tube noise was evident the assumption of a linear relationship between disturbance level and output noise permitted correction of the curve so that equivalent disturbance levels could be related to any setting of the noise attenuator, or to the receiver output noise level without feedback.

The signal-to-noise ratios at the output of the receiver without feedback are considerably higher than the corresponding ratios of carrier and disturbance levels existing at the amplitude detectors. This is the

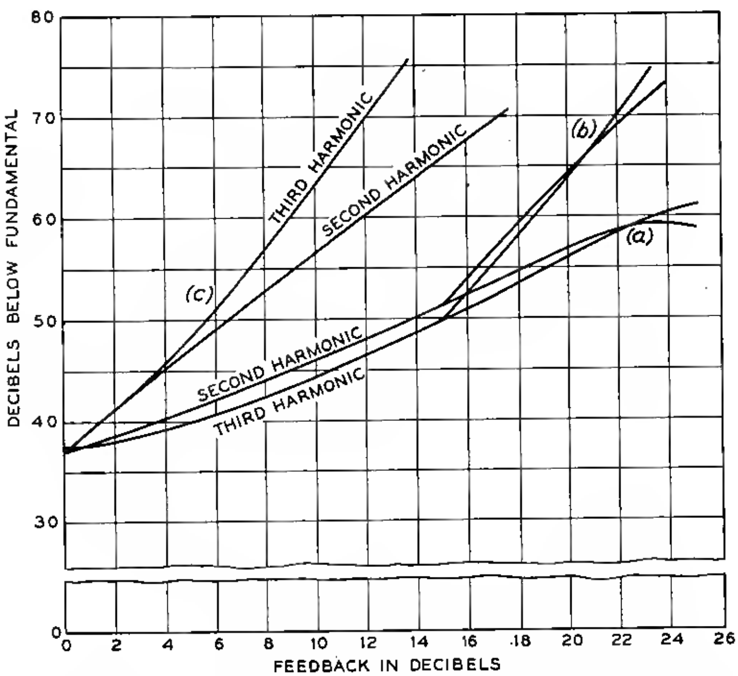

Fig. 12—Composite of data from Figs. 9 to 11 expressing ratios of harmonic levels to fundamental level. Curves (*a*) from Fig. 9, (*b*) from Fig. 10, and (*c*) from Fig. 11.

result of two factors. The intermediate-frequency wave is modulated to the extent of ± 7 kilocycles while the range of frequencies appearing at the output terminals of the receiver is limited to 4 kilocycles. Hence at the output of the balanced detector the noise level in the absence of modulation is 9.6 decibels below that which would be observed at the output of an amplitude-modulation system. Furthermore the admittance characteristic of the complete intermediate-frequency system is such that the effective disturbing voltage delivered to

the amplitude detectors is 11 decibels greater than that admitted by an amplitude modulation system having the minimum intermediate-frequency band width of 8 kilocycles.

Aural observation of the character of the output noise showed that, excluding the shaded area in Fig. 13, the normal characteristics of fluctuation noise are preserved as feedback is applied. Upon crossing

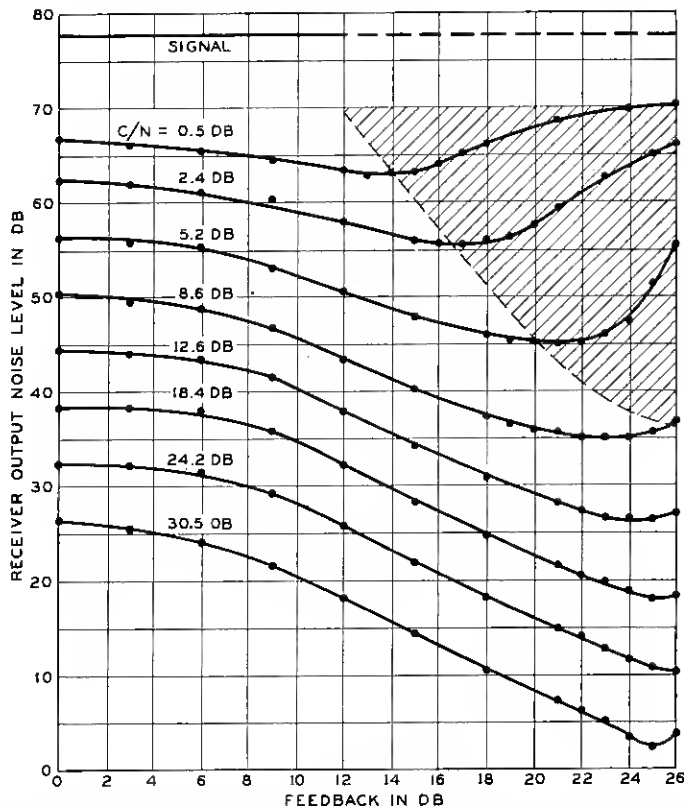

Fig. 13—Effect of feedback upon receiver output noise level for various amounts of high-frequency disturbance. Ratio of root-mean-square values of carrier and disturbance is shown on each curve. Shaded area indicates region of "crackling" in the absence of modulation.

the boundary of the shaded area the noise becomes punctuated with intermittent clicks which increase in rapidity and violence as the feedback factor is raised, giving rise to what can be described as "crackling." After passing through a region of maximum turbulence the noise gradually assumes the nature of a much higher level of fluctuation noise.

The region embracing the appearance of the above phenomenon is also characterized by a marked reduction in signal level. At high modulation levels "cracking" begins at a somewhat lower disturbance level than is necessary to initiate it in the absence of modulation. The initial effect is to impart a roughness to tone modulation. Further increase in disturbance level produces a rapid depression of the signal, so that it soon becomes submerged in noise. The manner in which this depression takes place is shown in Fig. 14. The signal, produced by 1000-cycle modulation was measured by means of a highly selective

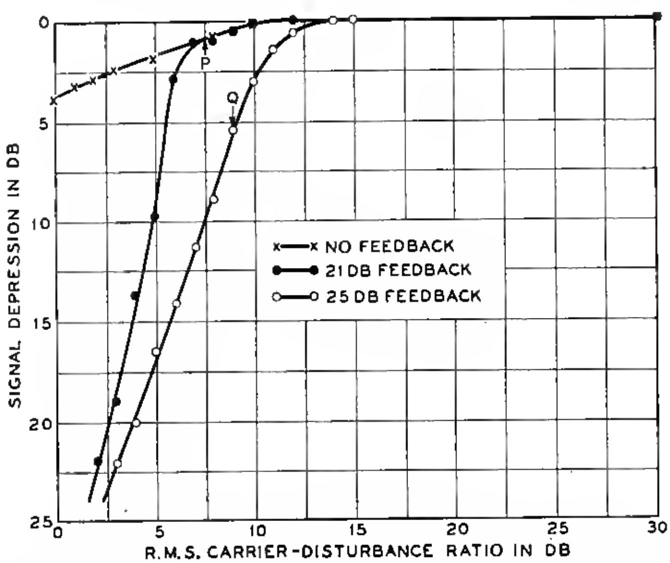

Fig. 14—Depression of output signal by the disturbance. Modulation: ± 7 kilocycles for no feedback, ± 78.5 kilocycles for 21-decibel feedback, and ± 124.5 kilocycles for 25-decibel feedback. Points *P* and *Q* denote incidence of crackling in the absence of modulation, for 21- and 25-decibel feedback, respectively.

analyzer so that observations could be carried well below the general noise level.

The point at which depression of the signal begins coincides with the appearance of roughness in the output tone resulting from the momentary suppression of the signal by the higher noise peaks. A further increase in disturbance level increases the number of peaks per second which rise above the critical value, and the energy content of the signal is rapidly diminished. The point at which faint crackling could first be detected in the absence of modulation is indicated on each curve.

The signal-to-noise ratios obtained with zero and with 25 decibels of

feedback are shown in Fig. 15. These are plotted against the ratio of root-mean-square carrier and disturbance levels at the end of the intermediate-frequency channel. Signal levels were measured in the presence of the disturbance so as to take into account the depression of the signal, while noise levels were observed in the absence of modulation.

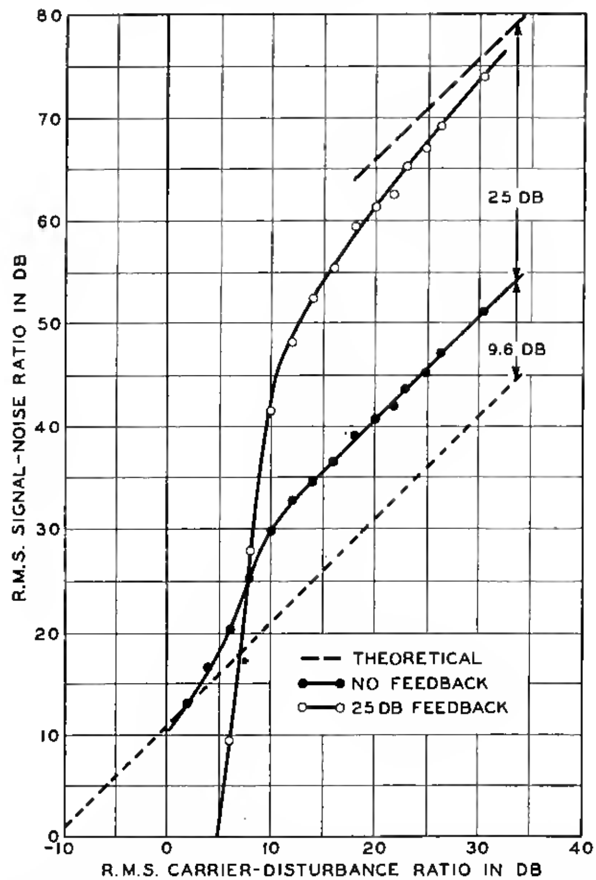

Fig. 15—Output signal-to-noise ratio vs. carrier-disturbance ratio with and without feedback. Modulation ± 124.5 kilocycles for 25-decibel feedback and ± 7 kilocycles for no feedback.

The improvement resulting from the application of feedback is given by the difference between the two curves and approaches the theoretical improvement at the low noise levels. The curve obtained with feedback exhibits a rather sharp break when the ratio of carrier to effective disturbance is in the vicinity of 10 decibels. Experimental

data [6] published by Crosby indicates that in the case of fluctuation noise the ratio of the maximum peak amplitude to the root-mean-square value is about 13 decibels. The corresponding figure in the case of a sine wave is 3 decibels. Hence equality of carrier peak amplitude and the highest peaks of the disturbance obtains when the ratio of their root-mean-square values is 10 decibels. With feedback this condition appears to define a fairly critical disturbance level above which the output signal-to-noise ratio is very rapidly diminished. Crosby has shown [6] that with systems employing amplitude limiters a similar condition marks the point beyond which the noise improvement realized at the lower disturbance levels is soon lost. This point he has termed the "threshold of noise improvement."

A less sharply defined break also occurs in the curve expressing noise conditions in the absence of feedback. This is the result of a progressive destruction, at the higher disturbance levels, of the balancing out of amplitude effects in the push-pull detector which is realized when the noise is low.

While direct comparison of the feedback system with an actual amplitude modulation system was not possible with the equipment used, it is thought that a comparison based upon theoretical considerations may be of interest. The procedure is as follows: The noise ratios shown in Fig. 15 for the system without feedback are, for disturbances below the threshold value, 9.6 decibels in excess of those which would be realized in a fully modulated amplitude system. A dotted line, displaced from the linear portion of the measured curve by this amount, is shown. The abscissae of the dotted curve do not represent the true carrier-disturbance ratio which would obtain in the amplitude system for the reason that, ideally, the intermediate-frequency amplifying system would have a band width of but 8 kilocycles. In such a system the signal-to-noise ratio would be equal to the carrier-disturbance ratio except at the very high noise levels. Hence the intercept of the dotted line with the axis of abscissas marks the point of equal carrier and disturbance levels in this system. The difference of 11 decibels between this point and the zero point on the scale as drawn measures the amount by which the disturbance at the rectifiers in the experimental system exceeds that which would be found in the ideal amplitude system.[8] Consequently, if it is desired to relate the data of Fig. 15 to the disturbance ratio which would exist at the input to the detector in the amplitude system, and hence to the signal-to-noise ratio in that

[8] Comparison of the areas under idealized curves representing the square of the transmission through the intermediate-frequency systems in the two cases indicates a difference of 10.1 decibels.

system, it is merely necessary to displace the experimental curves to the right by 11 decibels.

Figure 16 shows such a comparison between the feedback system adjusted to give 25 decibels of feedback, and an ideal amplitude modula-

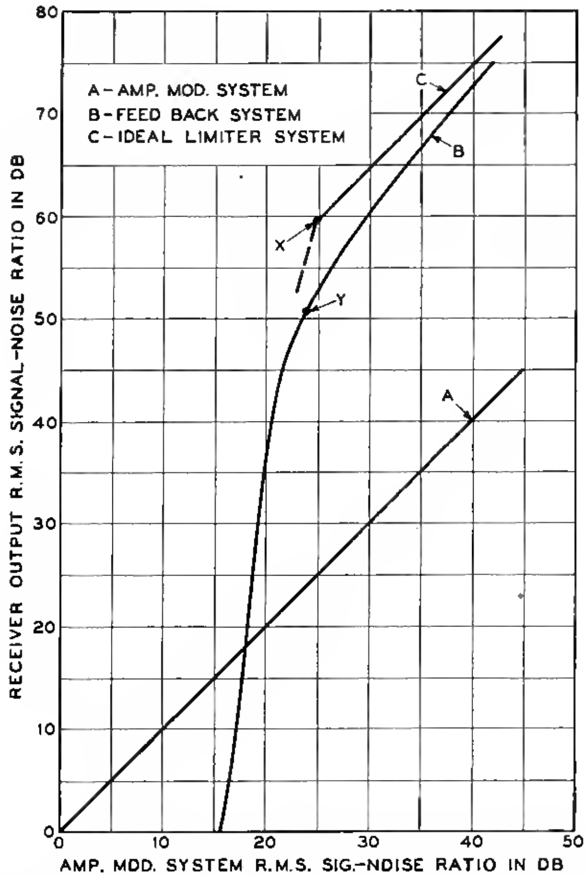

Fig. 16—Theoretical comparison of signal-to-noise ratios obtained at 25-decibel feedback (curve B) with amplitude-modulated system (curve A) and ideal limiter system with deviation ratio = 31.1 (curve C). Point X = threshold of noise improvement for limiter system. Point Y = point where "crackling" first became evident in the presence of $\pm$ 124.5-kilocycle modulation.

tion system. There is also included a curve showing the theoretical performance which would be approached by a frequency-modulation system using amplitude limitation. Transmitted band width and audio-frequency response equal to that used in the experimental feed-

back system have been assumed. This corresponds to a deviation ratio of 124.5 kilocycles ÷ 4 kilocycles = 31.1, resulting in a theoretical noise deduction of 34.6 decibels at low disturbance levels. The threshold of noise improvement, indicated at the point x, is located at a point where the peak signal-to-noise ratio in the amplitude system is equal to the square root of the deviation ratio.[6] This factor takes account of the higher disturbance level in the intermediate-frequency channel of the wide-band system. Assuming a factor of 10 decibels between maximum peak and root-mean-square values of fluctuation noise this corresponds to a root-mean-square signal-to-noise ratio of 24.9 decibels in the amplitude system.[9] In the feedback system the point at which crackling was first observed in the presence of modulation is shown at Y. This coincides very closely with the theoretical threshold of noise improvement in the limiter system.

Conclusions

It has been shown that the application of negative feedback to a frequency-modulation system affords a means for effecting large reductions in both noise and receiver distortion. The theoretical analyses of these effects, while they have been simplified to an extent which makes them inadequate to cover all conditions which can be encountered in practice, are adequately substantiated by the observed performance of the experimental system within the limitations of the theory.

Substantial benefits are realized only when the amount of feedback is large, and when the disturbance level is not too great. In common with frequency-modulation systems employing amplitude limitation a large reduction in noise must be paid for by increasing the band width of the transmitted wave. While the principles involved in the two systems are quite different, their performance as regards noise modification, both at high and at low levels of disturbance, exhibits striking similarities. The ability to reduce distortion is, on the other hand, a feature found only in the feedback system.

Acknowledgment

The writer wishes to acknowledge his indebtedness to the following of his colleagues: To Dr. H. W. Bode for his investigations of the problem of stability in feedback systems, and to Mr. R. L. Dietzold for the design of the stabilizing network which was used; to Mr. W. R. Bennett whose unpublished work on basic frequency-modulation prob-

[9] In the absence of more exact information regarding the performance of a system using this large a deviation ratio the portion of this curve below the threshold has been omitted.

lems has been of great value; and to Messrs. E. A. Krauth and O. E. DeLange for assistance in the experimental work. Reference has already been made to the theoretical work of Dr. J. R. Carson.

APPENDIX A

Analysis of Distortion Reduction

Assume that the transmitter is frequency-modulated with a signal wave which we shall represent by the symbol $S = S(t)$. Then the instantaneous frequency of the transmitter will be

$$\omega_1 + \rho_1 S. \tag{30}$$

The instantaneous phase of the transmitted wave is the integral of this expression and the voltage delivered to the input of receiver can be written

$$A \cos\left(\omega_1 t + \rho_1 \int_0^t S dt + \phi_1\right). \tag{31}$$

Designating the low-frequency voltage delivered at the output of the receiver as $\sigma = \sigma(t)$ the result of feeding back a portion of $k\sigma$ of the output so as to frequency-modulate the local oscillator is the wave

$$B \cos\left(\omega_2 t + \rho_2 \int_0^t k\sigma dt + \phi_2\right). \tag{32}$$

Application of these two waves to the modulator produces the intermediate-frequency product [10]

$$\alpha AB \cos\left[\omega_0 t + \rho_1 \int_0^t S dt - \rho_2 \int_0^t k\sigma dt + \phi_0\right] \tag{33}$$

where

$$\omega_0 = \omega_1 - \omega_2$$
$$\phi_0 = \phi_1 - \phi_2.$$

Terms in the above which involve the integral sign represent phase angles which vary with time. Hence we shall rewrite (33) more compactly

$$\alpha AB \cos\left[\omega_0 t + \theta(t) + \phi_0\right]. \tag{34}$$

It has been shown by Carson and Fry [3] that the process of detecting a frequency-modulated wave is, in effect, its differentiation. Since the high-frequency wave itself exhibits the integral of the signal wave, see

[10] This expression constitutes a more general form of (4).

(30), it can be reasoned that a differentiation process is necessary for the recovery of the signal itself.

Differentiation of the argument of the cosine term in (31) yields the instantaneous frequency (rate of change of phase with respect to time) of the received wave given by (30). Now it can be shown that with a strictly linear frequency detector, the low-frequency output is proportional to the response of the conversion system at the instantaneous frequency. The recovered signal is, therefore, proportional to the variable part of the instantaneous frequency and hence to the time derivative of the variable phase term in the original wave.

In the case of non-linearity in the characteristic of the frequency detector the output can be expressed, to a sufficiently close degree of approximation, as a power series in the derivative of the phase term $\theta(t)$. Hence the output of the receiver can be written in the form

$$\sigma(t) = \alpha AB \sum_n b_n \left[\frac{d}{dt} \theta(t) \right]^n \tag{35}$$

$$= \alpha AB \sum_n b_n [\rho_1 S - k\rho_2 \sigma]^n \tag{36}$$

where the coefficients b_n are based upon the transfer admittance characteristic of the receiver.

What is now desired is the relationship between σ and S. This can be expressed in the general form

$$\sigma = \alpha AB \sum_n c_n [\rho_1 S]^n. \tag{37}$$

Equations (36) and (37) can now be equated. Replacing σ in the right-hand side of (36) by the series (37) we shall have

$$\begin{aligned} c_1\rho_1 S + c_2(\rho_1 S)^2 + c_3(\rho_1 S)^3 + \cdots \\ = b_1[\rho_1 S - k\alpha AB\rho_2(c_1\rho_1 S + c_2(\rho_1 S)^2 + c_3(\rho_1 S)^3 + \cdots)] \\ + b_2[\rho_1 S - k\alpha AB\rho_2(c_1\rho_1 S + c_2(\rho_1 S)^2 + c_3(\rho_1 S)^3 + \cdots)]^2 \\ + b_3[\rho_1 S - k\alpha AB\rho_2(c_1\rho_1 S + c_2(\rho_1 S)^2 + c_3(\rho_1 S)^3 + \cdots)]^3 \\ + \cdots. \end{aligned} \tag{38}$$

After expanding, coefficients of like powers of $\rho_1 S$ can be equated. Then solving for the first three orders of c_n we find

$$c_1 = \frac{b_1}{1 + \alpha b_1 kAB\rho_2} = \frac{b_1}{1 - \mu\beta} \tag{39}$$

$$c_2 = \frac{b_2}{(1 - \mu\beta)^3} \tag{40}$$

$$c_3 = \frac{b_3}{(1 - \mu\beta)^4} - \frac{2\alpha ABk\rho_2 b_2^2}{(1 - \mu\beta)^5} \tag{41}$$

where

$$\mu = \alpha ABb_1 \qquad \text{and} \qquad \beta = -k\rho_2.$$

Inserting these values in (37) and writing $(1 - \mu\beta) = F$, the receiver output becomes, with feedback,

$$\sigma_F = \alpha AB\left[\frac{b_1}{F}\rho_1 S + \frac{b_2}{F^3}(\rho_1 S)^2 + \left(\frac{b_3}{F^4} - \frac{2b_2^2}{b_1}\cdot\frac{F-1}{F^5}\right)(\rho_1 S)^3\right]. \tag{42}$$

When $F \gg 1$ this can be written

$$\sigma_F = \alpha AB\left[\frac{b_1}{F}\rho_1 S + \frac{b_2}{F^3}(\rho_1 S)^2 + \frac{1}{F^4}\left(b_3 - \frac{2b_2^2}{b_1}\right)(\rho_1 S)^3\right]. \tag{43}$$

Without feedback we have

$$\sigma = \alpha AB[b_1\rho_1 S + b_2(\rho_1 S)^2 + b_3(\rho_1 S)^3]. \tag{44}$$

APPENDIX B

Analysis of Noise Reduction

In the following analysis it is assumed that the amplitude of the disturbance producing the noise is sufficiently small compared with that of the incoming signal wave so that the principle of superposition will apply. Hence the manner in which the effect of a single disturbing component is modified by feedback will first be developed. Then the effect of a disturbance consisting of a continuous spectrum is derived by direct summation.

Consider first the case where there are impressed upon the grid of the modulator the incoming wave and the local oscillator voltage [11] as defined by (31) and (32), plus a single disturbing component

$$Q\cos[(\omega_1 + \omega_n)t + \phi_n]. \tag{45}$$

Then the intermediate-frequency product will be

$$\alpha AB\cos\left[\omega_0 t + \rho_1\int_0^t S\,dt - \rho_2\int_0^t k\sigma\,dt\right] + \alpha BQ\cos\left[(\omega_0 + \omega_n)t - \rho_2\int_0^t k\sigma\,dt + \phi_n\right]. \tag{46}$$

For simplicity assume that the intermediate-frequency amplifier and conversion circuit have the ideal transfer admittance characteristic

$$Y(\omega) = a_0 + a_1(\omega - \omega_0). \tag{47}$$

[11] Arbitrary phase constants will be omitted from these expressions since they do not affect the final result.

Then all derivatives of Y with respect to ω above the first are zero, and the steady-state response is equal to its response at the instantaneous frequency of the applied wave. Hence after conversion we shall have

$$\alpha AB[a_0 + a_1(\rho_1 S - \rho_2 k\sigma)] \cos \left[\omega_0 t + \int_0^t (\rho_1 S - \rho_2 k\sigma) dt \right]$$
$$+ \alpha BQ[a_0 + a_1(\omega_n - \rho_2 k\sigma)] \cos \left[(\omega_0 + \omega_n)t - \int_0^t \rho_2 k\sigma dt + \phi_n \right]. \quad (48)$$

Application of (48) to a linear amplitude detector will yield a low-frequency output proportional to its amplitude. The amplitude factor is readily calculated for the case where $AB \gg BQ$. For if

$$X \cos x + Y \cos y = Z \cos z$$

then

$$Z = \sqrt{X^2 + Y^2 + 2XY \cos (x - y)}$$

and when $X \gg Y$

$$Z \doteq X + Y \cos (x - y). \quad (49)$$

Hence the output of the linear detector will be

$$\gamma \left(\alpha AB[a_0 + a_1(\rho_1 S - \rho_2 k\sigma)] + \alpha BQ[a_0 + a_1(\omega_n - \rho_2 k\sigma)] \times \cos \left[\omega_n t - \rho_1 \int_0^t S dt + \phi_n \right] \right). \quad (50)$$

The term $\alpha\gamma ABa_0$ represents direct current. Assuming that this is not fed back to the local oscillator we can then write

$$\sigma = A'[a_1(\rho_1 S - \rho_2 k\sigma)] + Q'[a_0 + a_1(\omega_n - \rho_2 k\sigma)] \cos \xi \quad (51)$$

where

$$A' = \alpha\gamma AB$$
$$Q' = \alpha\gamma BQ$$
$$\xi = \left(\omega_n t - \int_0^t \rho_1 S dt + \phi_n \right).$$

Solving for σ

$$\sigma = \frac{1}{1 + a_1 A' k\rho_2} \left[1 + \frac{a_1 Q' k\rho_2 \cos \xi}{1 + a_1 A' k\rho_2} \right]^{-1} \times [A' a_1 \rho_1 S + Q'(a_0 + a_1\omega_n) \cos \xi]. \quad (52)$$

If $Q' \ll A'$

$$\sigma \doteq \frac{1}{F}\left[1 - \frac{a_1 Q' k \rho_2 \cos \xi}{F}\right][A' a_1 \rho_1 S + Q'(a_0 + a_1 \omega_n) \cos \xi] \quad (53)$$

where $F = 1 + a_1 A' k \rho_2 = 1 - \mu\beta$ as before. Finally, neglecting terms in Q'^2, we have

$$\sigma = \frac{1}{F}\left[A' a_1 \rho_1 S + Q'\left(a_0 + a_1 \omega_n - \frac{A' a_1^2 \rho_1 k \rho_2 S}{F}\right) \cos \xi\right]. \quad (54)$$

The first term is the recovered signal while the remaining terms represent noise. Both signal and noise are modified by feedback. If we let

$$\rho_1 S = \Delta\omega \cos pt \quad (55)$$

then the noise becomes

$$\frac{Q'}{F}\left[(a_0 + a_1 \omega_n) - \frac{1}{F}(A' a_1^2 k \rho_2 \Delta\omega \cos pt)\right] \times \cos(\omega_n t - x \sin pt + \phi_n). \quad (56)$$

By means of the Jacobi expansions this can be put in the form

$$\frac{Q'}{F} \sum_{m=-\infty}^{\infty} \left[(a_0 + a_1 \omega_n) - \frac{A' a_1^2 k \rho_2 m p}{F}\right] J_m(x) \times \cos[(\omega_n - mp)t + \phi_n] \quad (57)$$

where $J_m(x)$ is the Bessel coefficient of the first kind.

Now let it be assumed that the disturbance consists of a very large number of sinusoidal components of like amplitude Q, random phase, and uniformly distributed along the frequency scale. The summation of this series of voltages can be represented by the very general expression

$$f(t) \cos[\omega t + \phi(t)]. \quad (58)$$

So long as $f(t)$, the equivalent amplitude of the high-frequency disturbance, is small compared with the carrier amplitude A, the approximation (49) will be valid and the total output noise can be obtained by summing up the effects of the individual elements which constitute the disturbance.

The effect of a single disturbing element is given by (57). Any term of this expression can be made to have the frequency q if m and ω_n are so chosen that

$$\omega_n = mp \pm q. \quad (59)$$

Then for each value of m in (57) there will be available values of ω_n

to satisfy both of the conditions expressed by (59). The total effect is obtained by summing the output power resulting from each contribution since the original elements have random phases. If r_2 is the resistance of the output circuit the total power of frequency q becomes

$$\frac{Q'^2}{2r_2F^2}\left(\sum_{m=-\infty}^{\infty}\left[a_0 + a_1(mp+q) - \frac{A'a_1^2kp_2mp}{F}\right]^2 J_m^2(x) + \sum_{m=-\infty}^{\infty}\left[a_0 + a_1(mp-q) - \frac{A'a_1^2kp_2mp}{F}\right]^2 J_m^2(x)\right). \quad (60)$$

This is readily evaluated with the aid of tables appended to an earlier paper.[12] The result is

$$\frac{Q'^2}{r_2F^2}\left[a_0^2 + \frac{a_1^2\Delta\omega^2}{2F^2} + a_1^2q^2\right]. \quad (61)$$

The amplitude factor Q remains to be defined. If N^2 is the mean disturbing power per unit band width in the vicinity of the carrier frequency and r_1 the resistance of the input circuit, the peak amplitude of any element is defined by the relation

$$N^2d\omega = \frac{Q^2}{2r_1}. \quad (62)$$

Thus the power associated with each element becomes differentially small, and if the value so obtained is entered into (61) there is obtained the output noise power contained in a band extending from q to $q + dq$. Then we shall have

$$dW = \frac{2N^2r_1(\alpha\gamma B)^2}{r_2F^2}\left[a_0^2 + \frac{a_1^2\Delta\omega^2}{2F^2} + a_1^2q^2\right]dq. \quad (63)$$

The total noise power in a band extending to a limiting frequency q_a is

$$P_n = \int_0^{q_a} dW = \frac{2N^2r_1(\alpha\gamma B)^2}{r_2F^2}\left[a_0^2 + \frac{a_1^2\Delta\omega^2}{2F_2} + \frac{a_1^2q_a^2}{3}\right]q_a. \quad (64)$$

The corresponding signal power is

$$P_s = \frac{(\alpha\gamma AB)^2}{2r_2F^2}a_1^2\Delta\omega^2. \quad (65)$$

[12] J. G. Chaffee, "The Detection of Frequency Modulated Waves," *Proc. I. R. E.*, vol. 23, pp. 517–540, May 1935.